DOCUMENT RESUME

ED 175 679 SE 028 667

AUTHOR Calloway, Jean M., Ed.
TITLE Supplementary and Enrichment Series: Systems of First Degree Equations in Three Variables. SP-21.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 65
NOTE 44p.; For related documents, see SE 028 648-675; Contains occasional light and broken type

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS *Algebra; *Analytic Geometry; Curriculum; *Enrichment; *Graphs; *Instruction; Mathematics Education; Secondary Education; *Secondary School Mathematics; Supplementary Reading Materials
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series is designed to make material for the study of topics of special interest to students readily accessible in classroom quantity. Topics covered include a three dimensional coordinate system, distance formula, the equation of a plane, first degree equations in three variables, systems of first degree equations in three variables, and the line of intersection of two intersecting planes. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED175679

SCHOOL MATHEMATICS STUDY GROUP

SP-21

SUPPLEMENTARY and ENRICHMENT SERIES

SYSTEMS OF FIRST DEGREE EQUATIONS IN THREE VARIABLES

Edited by Jean M. Calloway

U S DEPARTMENT OF HEALTH
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY.

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
SMSG
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SE 028 667

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

© 1965 by The Board of Trustees of the Leland Stanford Junior University
All rights reserved
Printed in the United States of America

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

Prepared under the supervision of the Panel on Supplementary Publications of the School Mathematics Study Group:

Professor R. D. Anderson, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana

Mr. Ronald J. Clark, Chairman, St. Paul's School, Concord, New Hampshire 03301

Dr. L. Eugene Ferguson, Newton High School, Newtonville, Massachusetts 02160

Mr. Thomas J. Hill, Montclair State College, Upper Montclair, New Jersey

Mr. Karl S. Kalman, Room 711D, Office of the Supt. of Schools, Parkway at 21st, Philadelphia 36, Pennsylvania 19103

Professor Augusta Schurrer, Department of Mathematics, State College of Iowa, Cedar Falls, Iowa

Dr. Henry W. Syer, Kent School, Kent, Connecticut

Professor Frank L. Wolf, Carleton College, Northfield, Minnesota 55057

Professor John E. Yarnelle, Department of Mathematics, Hanover College, Hanover, Indiana

CONTENTS

Section		Page
1.	A Three Dimensional Coordinate System	1
2.	Distance Formula in Three Dimensions	3
3.	An Equation of a Plane	5
4.	The Solution Set of an Equation in Three Variables	8
5.	The Graph of a First Degree Equation in Three Variables	10
6.	The Solution Set of a System of First Degree Equations in Three Variables	12
7.	The Solution Set of a System of Two First Degree Equations in Three Variables. Graphic Approach .	12
8.	Algebraic Representation of the Line of Intersection of Two Intersecting Planes	16
9.	The Solution Set of a System of Three First Degree Equations in Three Variables.	20
10.	Equivalent Systems of Equations in Three Variables	30
11.	Miscellaneous Exercises	34

SYSTEMS OF FIRST DEGREE EQUATIONS IN THREE VARIABLES

1. A Three Dimensional Coordinate System.

In this pamphlet we shall deal with triples of numbers (x,y,z) and view them sometimes as constituting solution sets of equations in three variables, and sometimes as representing points in three-dimensional space. Thus we will wish, at the outset, to set up a one-to-one correspondence between ordered triples of real numbers (x,y,z) and the points of three-dimensional space.

Take three mutually perpendicular lines and label these lines the x-axis, the y-axis, and the z-axis respectively. These lines can be chosen, and labelled, in any manner whatsoever. For the sake of uniformity, and because the choice is a common one, let the x- and y-axis be in a horizontal plane and the z-axis perpendicular to this plane. The point of intersection of the axes is O, the origin. We assign number scales to the axes, as we did with coordinate systems in one and two dimensions, in such a way that the zeros of each of the axes coincide at the origin. The positive direction $\overrightarrow{OX}$ extends forward, toward the observer; the positive direction $\overrightarrow{OY}$ extends to the right; and the positive direction $\overrightarrow{OZ}$ extends upward. A plane determined by any two of the axes is called a coordinate plane. There are three such planes, the XY-plane, the XZ-plane, and the YZ-plane.

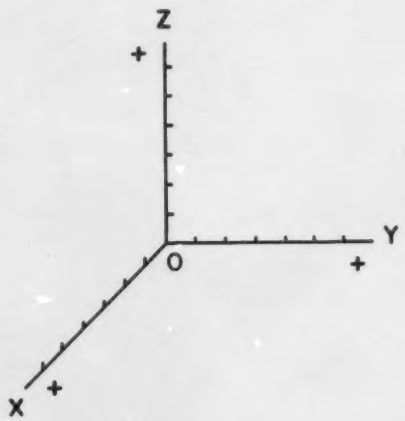


Figure 1a

Through any point P in space draw three planes which are respectively perpendicular to the three coordinate axes. The numbers attached to the points in which these planes intersect the x- , y- , and z-axes are called the x-coordinate, the y-coordinate, and the z-coordinate of the point P respectively. These planes and the three coordinate planes form a box-like figure (called a rectangular parallelepiped). We can then find the triple of coordinates of any given point in space; and, conversely, we can locate a point

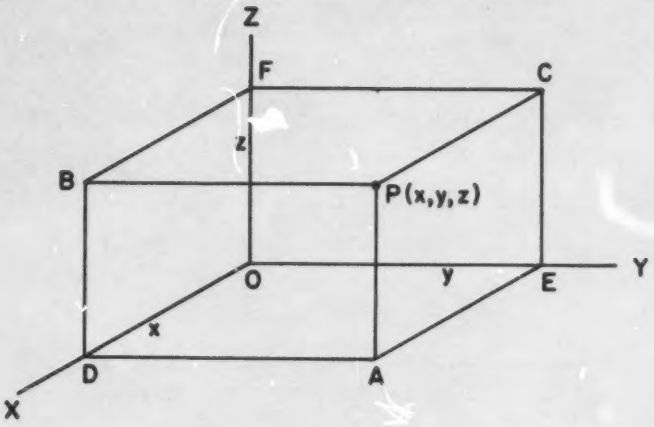


Figure 1b

in space when any ordered triple of real numbers is given. This one-to-one correspondence between points in space and the ordered triples of real numbers (x,y,z) is called a three-dimensional coordinate system.

Example: Plot the point (5,-2,4).

Solution: Begin at the origin and proceed 5 units in the direction of the positive x-axis, 2 units in the direction of the negative y-axis, and 4 units in the direction of the positive z-axis. The point located is the required point.

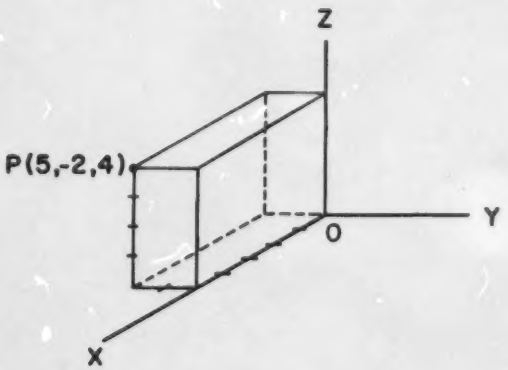


Figure 1c

Exercises 1.

Plot the following points:

1. (0, -1, 3)
2. (-2, 0, 4)
3. (3, 2, 4)
4. (2, -1, -3)
5. (-4, -2, -7)
6. (0, 2, 0)
7. (1, -1, 0)
8. (2, -3, 4)
9. (3, 2, -4)
10. (2, 0, -3)

11. Where do all points lie for which $x = 0$; for which $x = 2$; for which $x = -3$?
12. Where do all points lie for which $y = 0$; for which $y = 3$?
13. Where do all the points lie for which $z = 2$; for which $z = -2$?
*14. Where do all points lie for which $x + y = 4$?

---

2. Distance Formula in Three Dimensions.

Development of a formula for the distance between two points in space is closely related to the problem of finding the length of the diagonal of a rectangular parallelepiped. Let us review the latter problem first. By virtue of the Pythagorean relation we have

$$d^2(A,C) = d^2(A,D) + d^2(D,C)$$

$$d^2(A,B) = d^2(A,C) + d^2(C,B)$$

Substituting for $d^2(A,C)$ we have

$$d^2(A,B) = d^2(A,D) + d^2(D,C) + d^2(C,B)$$

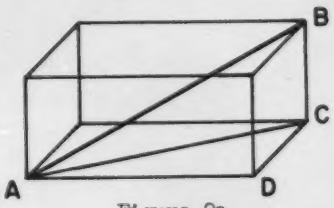


Figure 2a

$$(2a) \qquad d(A,B) = \sqrt{d^2(A,D) + d^2(D,C) + d^2(C,B)}$$

Thus, the diagonal of a rectangular parallelepiped equals the square root of the sum of the squares of its dimensions.

Consider now the distance between the points $A(1,2,4)$ and $B(3,5,6)$. These points are opposite vertices of a parallelepiped as indicated in Figure 2b. The distance between them, AB, may be obtained by applying formula

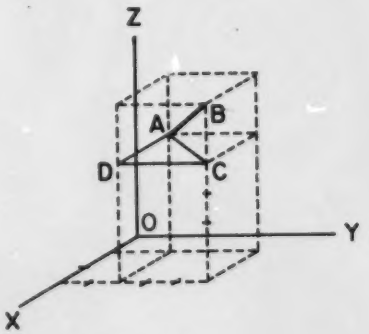


Figure 2b

(2a) $$d(A,B) = \sqrt{d^2(A,D) + d^2(D,C) + d^2(C,B)}.$$

From Figure 2b we see that

$$d(A,D) = 3 - 1 = 2$$
$$d(D,C) = 5 - 2 = 3$$
$$d(C,B) = 6 - 4 = 2$$
$$d(A,B) = \sqrt{4 + 9 + 4} = \sqrt{17}\ .$$

Using the same method, we now derive a formula for the distance between any two points in space, $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

From (2a) we have

$$d(P_1,P_2) = \sqrt{d^2(P_1,Q) + d^2(Q,R) + d^2(R,P_2)} \qquad \text{(See Fig. 2c.)}$$

But

$$d(P_1,Q) = |x_2 - x_1|$$
$$d(Q,R) = |y_2 - y_1|$$
$$d(R,P_2) = |z_2 - z_1|$$

(2b) $$\therefore\ d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

This is the formula for the distance between two points in three dimensions. The formula is correct no matter where P_1 and P_2 lie in space.

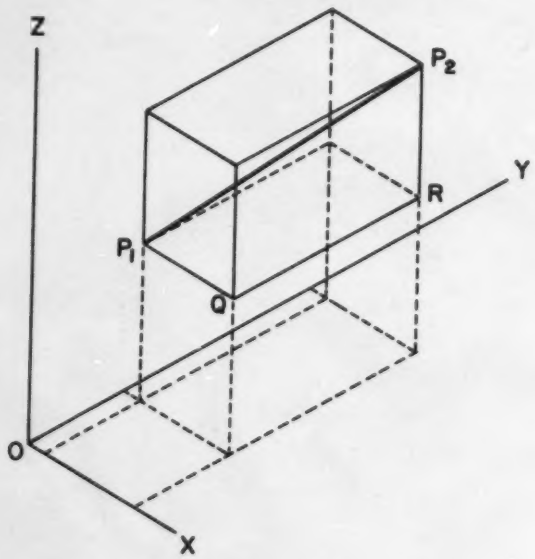


Figure 2c

Exercises 2.

Find the distance between the following pairs of points:

1. (6, 7, 1) , (2, 3, 1)
2. (4, -1, -5) , (7, 3, 7)
3. (0, 4, 5) , (-6, 2, 8)
4. (3, 0, 7) , (-1, 3, 7)
5. (4, -1, 3) , (12, 7, -1)
6. (-4, 2, -7) , (8, 18, 14)
7. (0, 1, 0) , (-1, -1, -2)
8. (-3, 4, -8) , (-8, -6, -6)
9. (3, 4, 5) , (8, 4, 1)
10. (1, 2, 3) , (0, 0, 0)

---

3. An Equation of a Plane.

From plane geometry we know that the set of points in a plane, at equal distances from two given points, is a line. Similarly, in space, the set of points at equal distances from two given points is a plane. We use this property to derive the equation of a plane. Since it was proved in geometry that this property characterizes a plane, the equation we derive will represent a plane with all the properties of the plane studied in geometry.

Example 1: Determine the equation of the plane whose points are equidistant from $A(1,2,3)$ and $B(2,5,4)$.

Solution: If $P(x,y,z)$ is any point in the plane, we know that

$$d(A,P) = d(B,P)$$

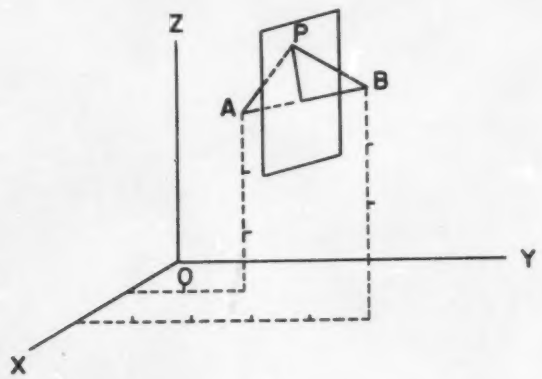


Figure 3a

Using Formula (2b), we have

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y-5)^2 + (z-4)^2}\,.$$

From this we have

$$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9$$

$$= x^2 - 4x + 4 + y^2 - 10y + 25 + z^2 - 8y + 16$$

which reduces to

(3a) $$2x + 6y + 2z = 31 .$$

Thus, the equation of this plane is of the first degree in 3 variables.

Using this same method we prove that the equation of every plane is an equation of first degree in 3 variables. Instead of two special points, A and B, we use $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$ to represent any two distinct points in space. Then we have

$$d(P_1,P) = d(P_2,P)$$

$$\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2}$$

$$x^2 - 2 \cdot x \cdot x_1 + x_1^2 + y^2 - 2 \cdot y \cdot y_1 + y_1^2 + z^2 - 2 \cdot z \cdot z_1 + z_1^2$$

$$= x^2 - 2 \cdot x \cdot x_2 + x_2^2 + y^2 - 2 \cdot y \cdot y_2^2 + y_2^2 + z^2 - 2 \cdot z \cdot z_2 + z_2^2$$

(3b)

$$2(x_2 - x_1)x + 2(y_2 - y_1)y + 2(z_2 - z_1)z - [(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)] = 0 .$$

Since $d(P_1,P)$ and $d(P_2,P)$ are positive numbers, this argument can be reversed. Therefore, we know that a point $P(x,y,z)$ whose coordinates satisfy equation (3b) is equidistant from P_1 and P_2.

Equation (3b) is an equation of first degree provided the coefficients of x, y, and z are not all zero. Let us denote these coefficients by

$$A = 2(x_2 - x_1) , \quad B = 2(y_2 - y_1) , \quad C = 2(z_2 - z_1) .$$

These will all be zero only if $x_2 = x_1$, $y_2 = y_1$, and $z_2 = z_1$, i.e., points P_1 and P_2 coincide. But P_1 and P_2 are distinct. Therefore, we have proved that every plane in three dimensions can be represented by an equation of the form

$$Ax + By + Cz + D = 0$$

where

$$D = -((x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2))$$

and A, B, C are real constants, not all zero. The converse theorem can also be proved, i.e., that every equation of this form represents a plane. The proof of this converse is given below.

Proof: Let $P(x,y,z)$ be any point on the plane that is the set of points equidistant from $O(0,0,0)$ and $Q(kA,kB,kC)$ where

$$k = \frac{-2D}{A^2 + B^2 + C^2}$$

Then

$$d(O,P) = d(Q,P)$$

$$x^2 + y^2 + z^2 = (x - kA)^2 + (y - kB)^2 + (z - kC)^2$$

$$0 = -2kAx + k^2A^2 - 2kBy + k^2B^2 - 2kCz + k^2C^2$$

$$2k(Ax + By + Cz) = k^2(A^2 + B^2 + C^2)$$

$$Ax + By + Cz = \frac{k}{2}(A^2 + B^2 + C^2) .$$

Put

$$k = \frac{-2D}{A^2 + B^2 + C^2} .$$

The equation becomes

$$Ax + By + Cz + D = 0.$$

This argument is reversible. This means that any point P whose coordinates satisfy $Ax + By + Cz + D = 0$ is equidistant from the two points O and Q. Hence $Ax + By + Cz + D = 0$ is, by definition, the equation of a plane.

Note: If $D = 0$, it follows that $k = 0$. The two points coincide, and no plane is determined. The case where $D = 0$ is treated in Problem 3, Exercise 3.

Exercises 3.

1. Use the method of Example 1 to find the equation of the plane whose points are equidistant from each of the following pairs of points:
   (a) (-1, 3, 2), (4, -2, -2); (d) (2, 4, -5), (0, 2, 3);
   (b) (-1, -3, -2), (-2, 0, 4); (e) (-2, 0, 6), (1, 4, 3);
   (c) (5, -1, 2), (-5, 1, -2); (f) (-1, 2, -3), (1, -2, 3).

2. In each of the following, find the equation of the plane that is the set of points equidistant from the given points, and sketch the graph.

(a) (4, 0, 0) , (-2, 0, 0)

(b) (0, 3, 0) , (0, -1, 0)

(c) (0, 0, 0) , (4, 2, 0)

(d) (0, 0, 0) , (0, 5, 3)

* 3. Prove that the equation

$$ax + by + cz = 0$$

where not all the constants a, b, c are zero, represents the set of points equidistant from the symmetric points (a,b,c) and (-a,-b,-c) .

---

4. The Solution Set of An Equation in Three Variables.

We shall examine several first degree equations in three variables, both graphically and algebraically, to gain familiarity with this representation of a plane.

Definition 4a. The solution set of an equation in three variables is the set of real number triples (x,y,z) that satisfy the equation.

Example 1: Find some of the elements of the solution set of the equation

(4a) $$x + 2y + z = 5 .$$

Solution: We may tabulate elements of the solution set of this equation by assigning values to x and y , and computing the corresponding values of z . In this way we may find as many number triples of the solution set as we wish.

In the first lines of the tabulation given below, we give the assigned values of x and y ; in the third line we give the computed value of z .

x	0	1	-1	1	2	0
y	0	1	1	-1	0	2
z	5	2	4	6	3	1

$$\begin{cases} x \text{ arbitrary} \\ y \text{ arbitrary} \\ z = 5 - x - 2y \end{cases}$$

Example 2: By considering sets of points in the solution set of

$$x + y = 4 ,$$

sketch the graph of the equation.

Solution: Viewed as an equation in three variables, this equation has the form

(4b) $$x + y + 0 \cdot z = 4 .$$

Since the coefficient of z in this equation is zero, we are no longer free to assign values to x and y at random. For example, if $x = 1$, we must assign the value 3 to y. On the other hand, when $x = 1$ and $y = 3$, we are free to assign any value whatsoever to z. We know from the definition of the coordinates of a point $P(x,y,z)$ (See Figure 1b) that all the points for which $x = 1$ and $y = 3$ lie on the perpendicular to the XY-plane through the point $(1,3,0)$. Since all these points $(1,3,z)$ correspond to number triples in the solution set of Equation (4b) no matter what value z has, we see that this perpendicular line lies in the plane $x + y + 0 \cdot z = 4$ (Figure 4a). Similarly, all points $(2,2,z)$, $(3,1,z)$, $(4,0,z)$ lie in the plane. Continuing in this fashion, we see that the plane contains all the perpendiculars to the XY-plane that intersect the XY-plane in the line $x + y = 4$. Since all these lines lie in a plane perpendicular to the XY-plane, we see that the equation $x + y = 4$ represents a plane perpendicular to the XY-plane. Its line of intersection with the XY-plane has the equation $x + y = 4$ $(z = 0)$.

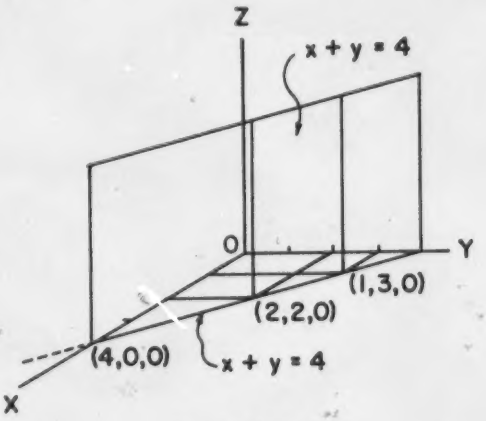


Figure 4a

Example 3: By considering subsets of the solution set of the equation $x = 3$, sketch a graph of the equation.

Solution: Viewed as an equation in three variables, this equation has the form

$$x + 0 \cdot y + 0 \cdot z = 3 .$$

Here x must be assigned the value 3, but y and z may assume any values. We see then that this plane is the set of points at the directed distance, $+3$, from the YZ-plane. It is therefore parallel to the YZ-plane.

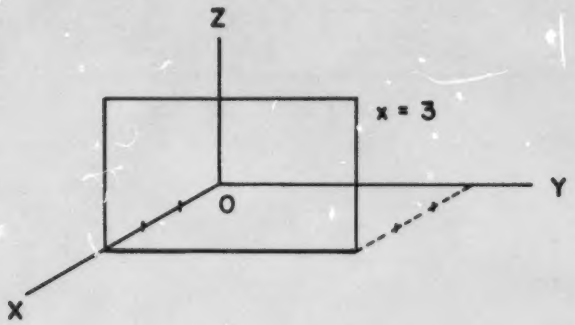


Figure 4b

Exercises 4.

1. Sketch the graphs of each of the equations
   (a) $x - 2y = 5$
   (b) $x - 2y = 0$
   (c) $y + 2z = 8$
   (d) $y - 2z = 0$
   (e) $2x - z = 0$
2. Four points on the graph of the plane

$$2x + y = 6$$

   are seen to be $A(3,0,0)$, $B(1,4,0)$, $C(2,2,0)$, $D(0,6,0)$. Give three other points on the graph with the same x and y values as A; as B; as C; as D. Sketch the graph.
3. Sketch the graph of $z = -2$; of $x = 5$; of $y = 3$.

---

5. The Graph of a First Degree Equation in Three Variables.

If either one or two of the coefficients in the equation

$$Ax + By + Cz + D = 0$$

are zero, Section 4 gives us a method of graphing the equation. If all the coefficients are different from zero, we proceed in a similar fashion.

Consider, for example, the graph of the equation

(5a) $$x + 2y + z = 5.$$

An easy way to plot the graph of a linear equation in the plane is to find the intercepts of the line. Similarly in three dimensions the graph of a plane is easy to sketch if we begin by finding the intersection of the plane with the coordinate planes. These intersections with the coordinate planes are called traces. If we want the intersection of plane (5a) with the

XY-plane we must put $z = 0$ in the equation

$$x + 2y + z = 5.$$

The resulting equation is

$$x + 2y = 5.$$

This is the equation of a striaght line in the XY-plane, and this straight line is called the trace of

$$x + 2y + z = 5$$

in the XY-plane. Similarly the XZ-trace is

$$x + z = 5,$$

and the YZ-trace is

$$2y + z = 5.$$

The graph of these lines in the coordinate planes makes the position of the plane

$$x + 2y + z = 5$$

clear.

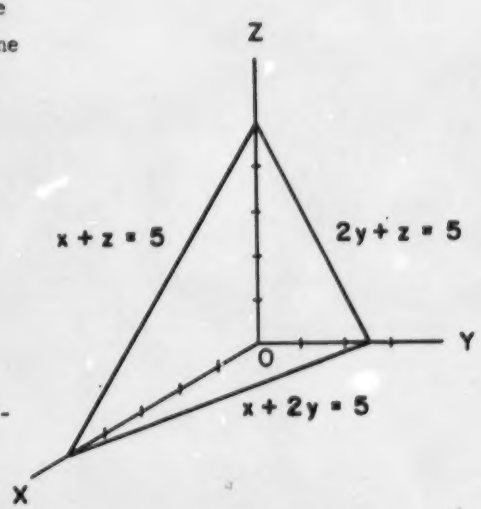


Figure 5

Exercises 5.

1. Sketch the graph of each of the following equations.
   (a) $x - 2y + z = 5$
   (b) $x + z = 5$
   (c) $x - 2y - z = 5$
   (d) $x + 2y + z = 5$
   (e) $4x - 2y + z = 0$
   (f) $5x + 4y = 20$
   (g) $3x - 2y + \frac{5}{3}z = 0$
   (h) $-\frac{x}{5} + \frac{y}{3} + \frac{z}{6} = 1$
   (i) $x - 2y - z = 0$
   (j) $2\frac{1}{2}x - 3\frac{1}{3}y = 0$

2. On the same set of axes sketch the graphs of the following pairs of equations, indicating the graph of the intersection set.

(a) $x + 2y + z = 5$
$x = 1$

(b) $x - 2y + z = 5$
$z = 2$

(c) $5x + 4y = 20$
$3x - 4y = 0$

(d) $5x + 4y = 20$
$-9x + 6y - 5z = 0$

(e) $x - 2y + z = 5$
$2x - 4y + 2z = 10$

---

6. The Solution Set of a System of First Degree Equations in Three Variables. Definitions.

Definition 6a. A system of first degree equations in three variables consists of two or more equations in three variables. In this pamphlet we will consider only systems that involve either two or three equations.

Definition 6b. The solution set of a system of first degree equations in three variables is the set of all number triples that satisfy all equations of the system. (It is the intersection of the solution sets of the equations of the system.)

Definition 6c. Two systems are equivalent if their solution sets are the same.

* 7. The Solution Set of a System of Two First Degree Equations in Three Variables. Graphic Approach.

In Section 3 we established the fact that every equation

$$Ax + By + Cz + D = 0$$

(in which A, B, and C are real coefficients not all zero) represents a plane. If we have two such first degree equations, they represent two planes that have one of three positions with respect to each other. The graphs of the two equations may intersect in a line, they may be parallel, or they may be the same plane. Our problem is to discuss the solution set of a system of two such equations. The most important case is the one in which the two planes intersect in a line. However, we will give an example to illustrate each of the three cases.

Example 1: The two planes intersect in a line. Find the solution set of the system

$$x + 2y + z - 5 = 0, \qquad x + z - 3 = 0. \tag{7a}$$

Solution: The complete solution set of the system (7a) may be obtained by studying the equivalent system obtained by combining either of the equations of (7a) with a combination

$$a(x + 2y + z - 5) + b(x + z - 3) = 0$$

of the equations of the system. By choosing $a = 1$, $b = -1$, we have

$$(x + 2y + z - 5) - (x + z - 3) = 0 ,$$

which reduces to

$$y = 1 . \tag{7b}$$

Thus, the line of intersection of the given planes, (7a), is also the line of intersection of the planes

$$\begin{cases} x + 2y + z = 5 \\ y = 1 \end{cases} \quad \text{or} \quad \begin{cases} x + z = 3 \\ y = 1 . \end{cases}$$

(1) The easiest system to graph is the last one.

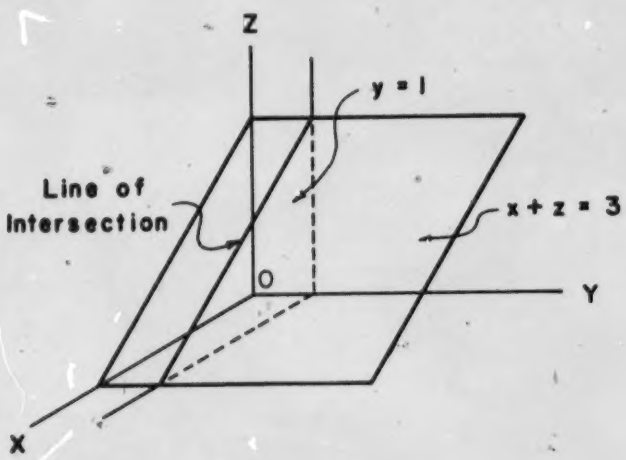


Figure 7a

[See also 7b and 7c.]

(2) Let us sketch the graph of the pair

$$x + 2y + z = 5,$$
$$y = 1.$$

The second plane is parallel to the XZ-plane and one unit to the right of it. Thus its trace in the XY-plane has a point of intersection with the XY-trace of the first plane; and its trace in the YZ-plane has a point of intersection with the YZ-trace of the first plane. Both these points have $y = 1$. They determine the line of intersection of the two planes. This line is parallel to the XZ-plane.

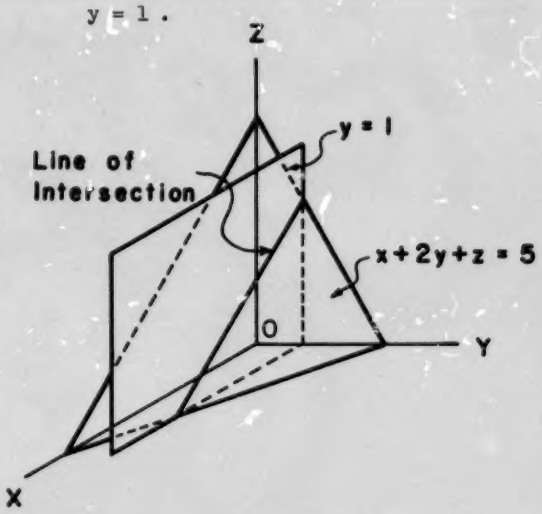


Figure 7b

(3) The third graph (Figure 7c) gives a sketch of the given planes

$$x + 2y + z = 5,$$
$$x + z = 3.$$

This graph is the most difficult to draw. The second plane has as its XY-trace the line

$$x = 3.$$

This intersects the XY-trace of the first plane, namely,

$$x + 2y = 5,$$

in the point $x = 3$, $y = 1$, $z = 0$. The traces of these two planes in the YZ-plane are

$$z = 3$$
$$2y + z = 5.$$

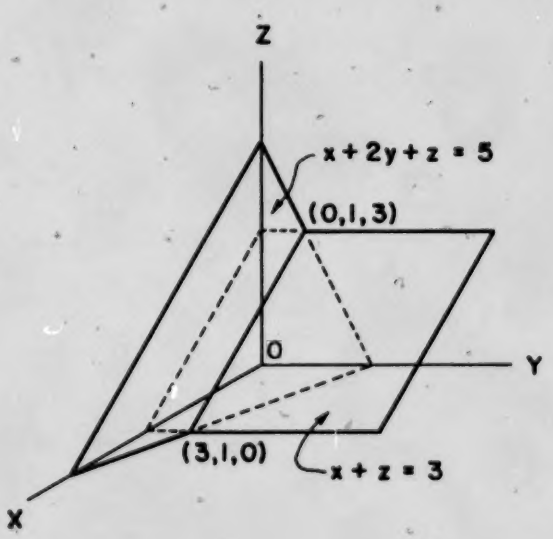


Figure 7c

They intersect in the point

$$x = 0 , y = 1 , z = 3.$$

We see that the line of intersection of these two planes is the same line as the one we obtained in (1) and (2), and that it is parallel to the XZ-plane.

Example 2. The two planes are parallel. Find the solution set of the system

$$x + 2y + z = 5,$$
$$x + 2y + z = 10.$$

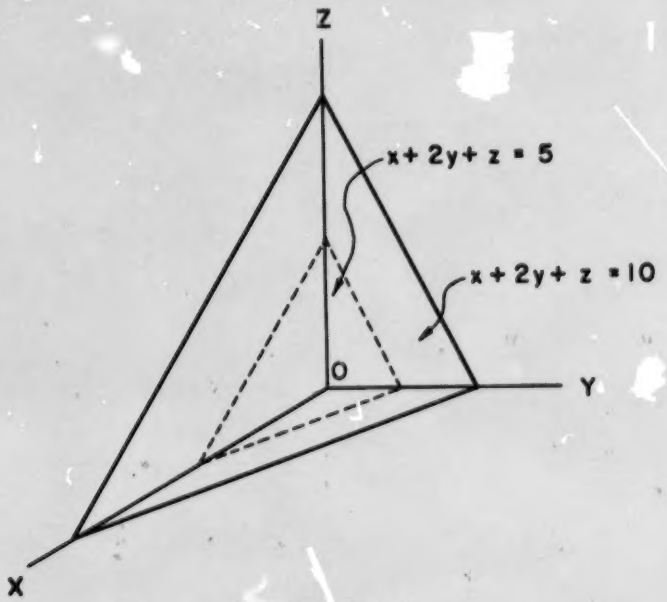


Figure 7d

Solution: By inspection, we can see that there is no number triple that satisfies both these equations. This is so because, for each number triple, the sum, $x + 2y + z$, has a definite value that cannot be both 5 and 10.

The planes have no point in common; they are parallel. The system is inconsistent, since any triple (x,y,z) that satisfies one equation will not satisfy the other.

Example 3. The planes coincide. Find the solution set of the system

$$x + 2y + z - 5 = 0,$$
$$3x + 6y + 3z - 15 = 0.$$

Solution: By inspection, we can see that every number triple in the solution set of the first equation is also in the solution set of the second equation; and conversely. The given planes coincide. The system is dependent; the left member of the second equation is three times the left member of the first equation.

Exercises 7.

Determine which of the following pairs of equations represent straight lines. Sketch the graph in each case. When the planes intersect, indicate on the graph where the line of intersection lies.

1. $x - 2y + 5z = 10$,
   $z = 1$.
2. $x - 2y + 5z = 10$,
   $x = 4$.
3. $x - 2y + 5z = 10$,
   $y = -2$.
4. $x + y = 5$,
   $x = 7 + y$.
5. $x + y = 5$,
   $x + y + z = 10$.
6. $3y + z = 9$,
   $x + 4y = 4$.
7. $x + 4y = 4$,
   $z - x = 0$.
8. $3x + y - z = 2$,
   $2z = 6x + 2y - 4$.
9. $z - x = 0$,
   $3y + z = 9$.
10. $x = -2$,
    $z = 4$.
11. $x + 2y + z = 5$,
    $-x + 2y + z = 5$.
12. $x + 2y + z = 8$,
    $x - 2y = 0$.

8. Algebraic Representation of the Line of Intersection of Two Intersecting Planes.

In this section we study the intersection of a pair of planes

(8a) $$x + 2y - z - 5 = 0$$

(8b) $$x + y + z - 2 = 0 .$$

Our procedure is to obtain the equations of three planes which pass through the line of intersection of the given planes and which give particularly useful representations of that line. We construct three different linear combinations of the expressions

$$(x + 2y - z - 5)$$

and

$$(x + y + z - 2) ,$$

and find three components of equivalent systems each of which has the coefficient of at least one variable equal to zero.

A component of an equivalent system can be written

$$a(x + 2y - z - 5) + b(x + y + z - 2) = 0.$$

(1) We eliminate x by choosing $a = 1$, $b = -1$.

$$(x + 2y - z - 5) - (x + y + z - 2) = 0$$

(8c) $$y - 2z - 3 = 0$$

(2) We eliminate y by choosing $a = 1$, $b = -2$.

$$(x + 2y - z - 5) - 2(x + y + z - 2) = 0$$

(8d) $$-x - 3z - 1 = 0$$

(3) We eliminate z by choosing $a = 1$, $b = 1$.

$$(x + 2y - z - 5) + (x + y + z - 2) = 0$$

(8e) $$2x + 3y - 7 = 0$$

We now have three distinct new equations (8c), (8d), (8e), any two of which may be chosen to represent the line of intersection of the given planes.

If we represent this line by the planes

(8c) $$y - 2z - 3 = 0$$

and

(8d) $$-x - 3z - 1 = 0$$

we can express x and y in terms of z:

(8f) (1) $$\begin{cases} x = -3z - 1, \\ y = 2z + 3. \end{cases}$$

This is an especially convenient form for determining particular points on the line of intersection of the two given planes. It enables us easily to write down as many number triples in the solution set as we wish. We see that we may assign values to z at random, and obtain corresponding values of x and y. Thus the solution set contains infinitely many number triples. This is what we should have expected, since the intersection of these two planes is a line.

Example 1. Write 4 members of the solution set of the above system (1).

Solution: Using the first representation given above, (8f), assign arbitrary values to z, and compute the corresponding values of x and y.

x	-1	-4	2	-7
y	3	5	1	7
z	0	1	-1	2

$$\begin{cases} x = -3z - 1 \\ y = 2z + 3 \\ z \text{ arbitrary} \end{cases}$$

If we use (8d) and (8e), we can express y and z in terms of x;

(8g)
$$y = -\frac{1}{3}(2x - 7),$$
$$z = -\frac{1}{3}(x + 1).$$

Using this representation of the line of intersection of the two given planes, check the number of triples obtained above by assigning the tabulated values of x, and computing the other values.

x	-1	-4	2	-7
y				
z				

$$\begin{cases} x \text{ arbitrary} \\ y = -\frac{1}{3}(2x - 7) \\ z = -\frac{1}{3}(x + 1) \end{cases}$$

Using (8c) and (8e) we can express x and z in terms of y :

(8h)
$$x = -\frac{1}{2}(3y - 7),$$
$$z = \frac{1}{2}(y - 3).$$

Using this representation, check again the number triples obtained from (8f) by assigning the tabulated values of y, and computing the corresponding values of x and z.

x				
y	3	5	1	7
z				

$$\begin{cases} x = -\frac{1}{2}(3y - 7) \\ y \text{ arbitrary} \\ z = \frac{1}{2}(y - 3) \end{cases}$$

Example 2: Find four number triples in the solution set of the system

$$2x - y + 2z = 6,$$
$$z = 2.$$

How can we describe the whole solution set algebraically?

Solution: In this example every number triple in the solution set has $z = 2$. By substituting this value in the first equation we have

$$2x - y = 2$$
$$y = 2x - 2$$

or $$x = \frac{1}{2}(y + 2) .$$

Thus four number triples in the solution set can be written by assigning arbitrary values to x, and computing the values of y:

x	0	1	-1	2
y	-2	0	-4	2
z	2	2	2	2

$$\begin{cases} x \text{ arbitrary} \\ y = 2x - 2 \\ z = 2 \end{cases}$$

or by assigning arbitrary values to y, and computing the values of x:

x				
y	-2	0	-4	2
z				

$$\begin{cases} x = \frac{1}{2}(y + 2) \\ y \text{ arbitrary} \\ z = 2 \end{cases}$$

The complete description of the solution set is given either

as $$\begin{cases} x \text{ arbitrary} \\ y = 2x - 2 \\ z = 2 \end{cases}$$ or as $$\begin{cases} x = \frac{1}{2}(y + 2) \\ y \text{ arbitrary} \\ z = 2 . \end{cases}$$

In this case, z may not be chosen arbitrarily.

Exercises 8.

In each of the problems given below if the planes intersect in a line express two of the variables of the solution set in terms of the third variable, and tabulate a subset of the solution set consisting of four number triples.

1. $x - 3y - z = 11$
   $x - 5y + z = 1$
2. $x + 2y - z = 8$
   $x + y + z = 0$
3. $x - z + y = 5$
   $x + 2y = -z$
4. $2x + 4y - 7 = 5z$
   $4x + 8y - 14 = 5z$
5. $-2x + y + 3z = 0$
   $-4x + 2y + 6z = 0$
6. $2x + 6z - 18y = 6$
   $x - 3z - y = -3$
7. $3x - 4y + 2z = 6$
   $6x - 8y + 4z = 14$
8. $-5x + 4y + 8z = 0$
   $-3x + 5y + 15z = 0$
9. $6z - 7y + 4x = 13$
   $5x + 6y - z = 7$
10. $-10x + 4y - 5z = 20$
    $2x - \frac{4}{5}y + z = 4$

9. The Solution Set of a System of Three First Degree Equations in Three Variables.

We now consider the solution set of three first degree equations in three variables. A simple example will introduce us to the problem.

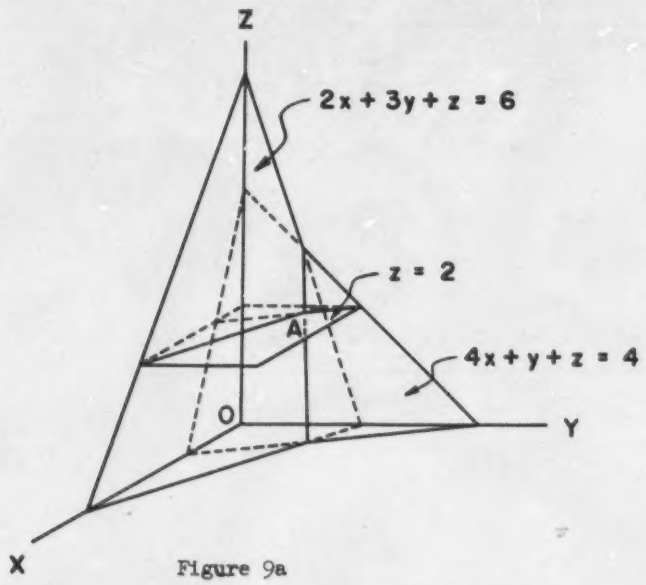


Figure 9a

$$2x + 3y + z = 6 ,$$
$$4x + y + z = 4 ,$$
$$z = 2 .$$

Figure 9a suggests the graphic solution in which A is the single point of intersection of the three planes. Algebraically, we may use the value of z given by the third equation; substitute it in the first two equations, and then solve for x and y :

$$2x + 3y = 4 ,$$
$$4x + y = 2 ;$$
$$x = \frac{1}{5} ; \; y = \frac{6}{5} ; \; z = 2 .$$

The point of intersection of the three planes is $(\frac{1}{5},\frac{6}{5},2)$.

Usually the graphic representation of the three planes represented by three first degree equations in three variables will be too complicated to draw. But it is helpful to keep in mind the geometric meaning of the equations when we consider the types of solution sets that we may expect. These correspond to the types of intersections that are possible for three planes

in space. The method of solution will be the same in all cases. It is illustrated by the following examples. In each case the problem is to find the solution set.

Example 1:

$$x + 2y - 3z = 9 ,$$
$$2x - y + 2z = -8 ,$$
$$-x + 3y - 4z = 15 .$$

Step 1: Eliminate x from the second and third equations by adding appropriate multiples of the first equation. We now have the equivalent system

$$x + 2y - 3z = 9 ,$$
$$0 - 5y + 8z = -26 ,$$
$$0 + 5y - 7z = 24 .$$

Step 2: Eliminate y from the third equation by adding an appropriate multiple of the second equation obtaining the equivalent system

$$x + 2y - 3z = 9 ,$$
$$0 - 5y + 8z = -26$$
$$0 + 0 + z = -2 .$$

Step 3: Substitute $z = -2$ in the second equation obtaining

$$-5y = -26 + 16$$
$$y = 2 .$$

Step 4: Substitute $z = -2$, and $y = 2$ in the first equation.

$$x + 4 + 6 = 9$$
$$x = -1 .$$

Step 5: Check the solution.

$$-1 + 4 + 6 = 9 ,$$
$$-2 - 2 - 4 = -8 ,$$
$$1 + 6 + 8 = 15 .$$

We see that the solution is the number triple $(-1,2,-2)$. The planes intersect in a point. Figure 9b, shows three planes intersecting in a point. (Case 1.)

Example 2:

$$2x - 3y + z - 3 = 0 ,$$
$$x + 5y - z - 3 = 0 ,$$
$$5x + 12y - 2z - 12 = 0 .$$

To simplify the arithmetic, we interchange the first and second equation, and proceed with the steps described in Example 1.

Step 1: Eliminate x from two equations, obtaining the equivalent system,

$$\begin{aligned} x + 5y - z - 3 &= 0 , \\ 0 - 13y + 3z + 3 &= 0 , \\ 0 - 13y + 3z + 3 &= 0 . \end{aligned}$$

Step 2: Eliminate y from the third equation.

$$\begin{aligned} x + 5y - z - 3 &= 0 , \\ 0 - 13y + 3z + 3 &= 0 , \\ 0 + 0 + 0 + 0 &= 0 . \end{aligned}$$

In this case, the third equation contributes no new information. If Step 2 gives the identity, $0 = 0$, one of the given equations is a linear combination of the other two. Here the left member of the third equation,

$$5x + 12y - 2z - 12$$

can be obtained as

$$(2x - 3y + z - 3) + 3(x + 5y - z - 3).$$

Therefore, we know that the graph of the third equation must pass through the line of intersection of the planes

$$\begin{aligned} 2x - 3y + z - 3 &= 0 \\ x + 5y - z - 3 &= 0, \end{aligned}$$

(This relationship will be studied further in Section 10.) Thus, the complete solution of the given system is an infinite set of triples representing the points on the line of intersection of the given planes. We may use the method in Section 8 if we wish to determine the numbers of the solution set.

Eliminating x from the first two equations we have

$$\begin{aligned} (2x - 3y + z - 3) - 2(x + 5y - z - 3) &= 0 \\ -13y + 3z + 3 &= 0 . \end{aligned}$$

Eliminating z from the first two equations we have

$$\begin{aligned} (2x - 3y + z - 3) + (x + 5y - z - 3) &= 0 \\ 3x + 2y - 6 &= 0 . \end{aligned}$$

Solving for x and z in terms of y :

$$x = \frac{1}{3}(-2y + 6) ,$$

$$z = \frac{1}{3}(13y - 3) ,$$

y is arbitrary.

Figure 9b shows three planes intersecting in a straight line. (Case 2a.)

Example 3:

$$x + 5y - z - 3 = 0 ,$$
$$2x - 3y + z - 3 = 0 ,$$
$$2x + 10y - 2z - 6 = 0 .$$

Here Step 1 yields

$$x + 5y - z - 3 = 0 ,$$
$$0 - 13y + 3z + 3 = 0 ,$$
$$0 + 0 + 0 + 0 = 0 .$$

We see that the left-hand member of the third equation is twice the left-hand member of the first equation.

$$2x + 10y - 2z - 6 = 2(x + 5y - z - 3).$$

Therefore, the first and third planes coincide. Again, the solution is completely described by the first two equations. It is the same line we found in Example 2. (See Case 2b, Figure 9b.)

Example 4:

$$x + 2y + z = 4,$$
$$x - 2y + z = 0 ,$$
$$x + z = 4 .$$

For simplicity, move the third equation into the first row

$$x + z = 4 ,$$
$$x + 2y + z = 4 ,$$
$$x - 2y + z = 0 .$$

Step 1: Eliminate x in the second and third equations.

$$x + 0 + z = 4 ,$$
$$0 + 2y + 0 = 0 ,$$
$$0 - 2y + 0 = -4 .$$

Step 2: Eliminate y from the new third equation.

$$x + 0 + z = 4 ,$$
$$0 + 2y + 0 = 0 ,$$
$$0 + 0 + 0 = -4 .$$

Since there are no triples for which $0 = -4$ there are no solutions. In this case one plane is parallel to the intersection of the other two. (See Case 4a, Figure 9b.)

Example 5:

$$x + y + 2z = 1 ,$$
$$x + y + 2z = 2 ,$$
$$x + y + 2z = 3 .$$

By subtracting the first equation from the other two we find immediately

$$x + y + 2z = 1 ,$$
$$0 + 0 + 0 = 1 ,$$
$$0 + 0 + 0 = 2 .$$

Again, we have no solution. The three planes are parallel. (See Case 4d, Figure 9b.)

Example 6:

$$x - y - 2z = 1 ,$$
$$2x - 2y - 4z = 2 ,$$
$$-x + y + 2z = -1 .$$

Step 1 gives the equivalent system

$$x - y - 2z = 1 ,$$
$$0 + 0 + 0 = 0 ,$$
$$0 + 0 + 0 = 0 .$$

In this case, the three equations represent the same plane. (See Case 3, Figure 9b.)

Example 7:

$$\frac{3}{x} + \frac{2}{y} - \frac{1}{z} = 3 ,$$
$$\frac{2}{x} + \frac{3}{y} + \frac{2}{z} = 3 ,$$
$$\frac{4}{x} + \frac{1}{y} - \frac{3}{z} = 4 .$$

These equations are linear in the variables $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$. We treat these reciprocals as the unknowns.

Retain the first equation, changing the order of the variables so that the computation that follows can be carried on more conveniently.

$$-\frac{1}{z} + \frac{2}{y} + \frac{3}{x} = 3$$

Step 1: Eliminate $\frac{1}{z}$ from the second and third equations. We have the equivalent system

$$-\frac{1}{z} + \frac{2}{y} + \frac{3}{x} = 3 ,$$
$$0 + \frac{7}{y} + \frac{8}{x} = 9 ,$$

$$0 + \frac{5}{y} + \frac{5}{x} = 5 \quad \text{or} \quad 0 + \frac{1}{y} + \frac{1}{x} = 1 .$$

Step 2: Eliminate $\frac{1}{y}$ from the third equation

$$-\frac{1}{z} + \frac{2}{y} + \frac{3}{x} = 3 ,$$
$$0 + \frac{7}{y} + \frac{8}{x} = 9 ,$$
$$0 + 0 + \frac{1}{x} = 2 . \qquad \therefore \; x = \frac{1}{2} .$$

Step 3: Substitute $\frac{1}{x} = 2$ in the second equation.

$$\frac{7}{y} + 16 = 9$$
$$\frac{7}{y} = -7$$
$$\frac{1}{y} = -1 \qquad \therefore \; y = -1 .$$

Step 4: Substitute $\frac{1}{x} = 2$, $\frac{1}{y} = -1$ in the first equation, obtaining

$$-\frac{1}{z} - 2 + 6 = 3$$
$$\frac{1}{z} = 1 . \qquad \therefore \; z = 1 .$$

Step 5: Check the solution.

$$3 \cdot 2 + 2(-1) - 1 = 3 ,$$
$$2 \cdot 2 + 3(-1) + 2 = 3 ,$$
$$4 \cdot 2 + (-1) - 3 = 4 .$$

Summary. The method described in this section is called triangulation because, in the case of a unique solution, the non-zero coefficients (represented by Step 2 in Example 1) lie in the form of a triangle:

1	2	-3
0	-5	8
0	0	1

This method provides a systematic procedure that enables us to recognize when the solution set is empty, when it contains a single triple, and when it contains infinitely many triples corresponding either to a line of points or to a plane of points. The method can be summarized as follows:

Step 1: After choosing a convenient first equation, eliminate one variable (say x) from the other two equations by adding appropriate multiples of the (chosen) first equation.

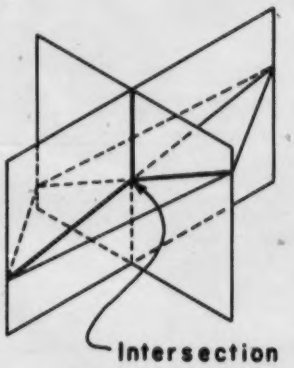
Step 2: In a similar way, work with the second and third equations which now contain only y and z. Multiplying by suitable numbers, eliminate a second variable (say y) from the third equation.

Steps 3 and 4: The third equation now gives a value of one variable (say z). Substitute this value in the second equation to obtain y. Substitute the values of y and z in the first equation to obtain x.

Step 5: Check the values of x, y, z found in Steps 1 - 4 in the given equations.

In Figure 9b we give sketches that illustrate the possible types of intersection of three planes in space.

1. The three planes intersect in a point. The solution set is a single number triple.



2. The three planes intersect in a line. The solution set is the infinite set of number triples corresponding to the points on the line.
   (a) The three planes have a line in common.

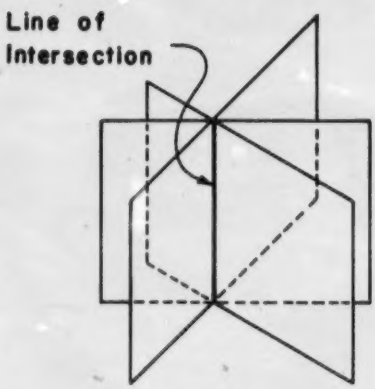
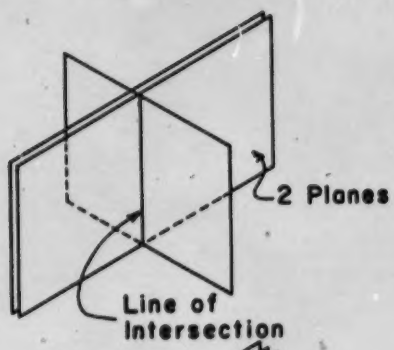
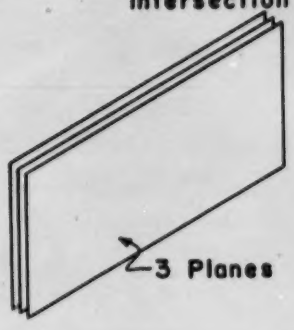


Figure 9b

2. (b) Two planes coincide and intersect the third plane in a line. The solution set is the same as in 2(a).

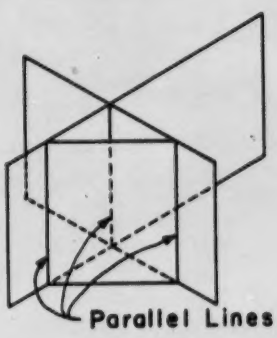


3. All three planes coincide. The solution set is the infinite set of number triples corresponding to the points in the plane.



4. The three planes do not have a common intersection. The solution set is empty. The system is inconsistent.

   (a) Two planes intersect; the third is parallel to their intersection.



   (b) Two planes are parallel. The third plane intersects these two in parallel lines.

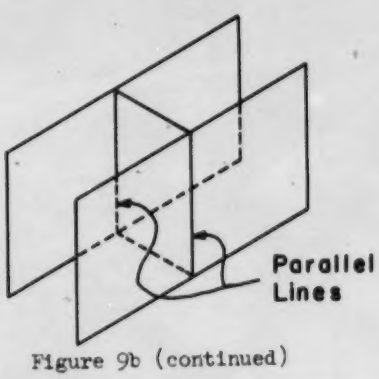
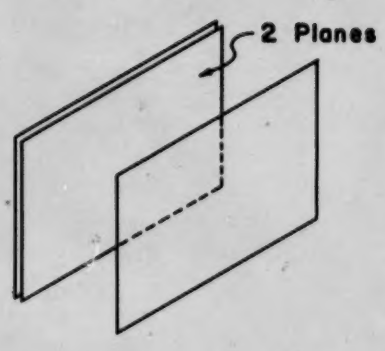


Figure 9b (continued)

4. (c) Two planes coincide and are parallel to the third plane.



(d) The three planes are parallel.

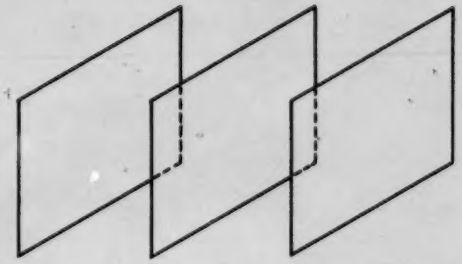


Figure 9b (continued)

Exercises 9.

In each of the following problems, determine whether the solution set is empty or whether its graph is a point, a line, or a plane. If the intersection is a point, give its coordinates.

1. $x + z = 8$,
   $x + y + 2z = 17$,
   $x + 2y + z = 16$.
2. $x + 2y - z = 5$,
   $x + y + 2z = 11$,
   $x + y + 3z = 14$.
3. $x + 2y - z = -1$,
   $2x + 2y - 3z = -1$,
   $4x - y + 2z = 11$.
4. $x + y - 5z = 9$,
   $2x + 3y - 12z = 22$,
   $3x - 5y + z = -5$.
5. $x - 2y + 3z = 6$,
   $2x + y - 2z = -1$,
   $3x - 3y - z = 5$.
6. $2x + 4y + z = 0$,
   $x - y + 3z = 8$,
   $3x + y - 2z = -2$.

7. $x - 2y + z = 4$,
$-3x + 6y - 3z = -12$,
$2x - 4y + 2z = 8$.

8. $2x + 3y + 7z - 13 = 0$,
$3x + 2y - 5z + 22 = 0$,
$5x + 7y - 3z + 28 = 0$.

9. $4x - y + z = 6$,
$3x + 2y - 4z = 2$,
$7x + y - 3z = 5$.

10. $20x - 20y - 30z = 0$,
$15x - 10y - 25z = 0$,
$10x - 20y - 10z = 0$.

11. $\frac{10}{3x} + \frac{3}{y} + \frac{2}{z} = 2$,
$\frac{10}{3x} + \frac{1}{y} - \frac{3}{z} = 0$,
$\frac{4}{x} - \frac{18}{y} + \frac{3}{z} = -1$.

12. $\frac{5}{x} + \frac{12}{y} + \frac{10}{z} = 1$,
$\frac{3}{x} + \frac{8}{y} - \frac{2}{z} = 1$,
$-\frac{4}{x} - \frac{8}{y} - \frac{3}{z} = -5$.

13. $x + y + z = 2$,
$2x + 2y + 2z = 5$,
$x - y + z = 7$.

14. $3x - y - 2z - 2 = 0$,
$2y - z + 1 = 0$,
$3x - 5y - 3 = 0$.

15. $\frac{3}{x} - \frac{1}{y} = 7$,
$\frac{3}{y} - \frac{1}{z} = 5$,
$\frac{3}{z} - \frac{1}{x} = 0$.

16. $x + y + z = 3$,
$3x + 3y + 3z = 9$,
$x + y - z = 6$.

17. $\frac{1}{x} + \frac{2}{y} - \frac{1}{z} = 5$,
$\frac{2}{x} - \frac{1}{y} + \frac{1}{z} = 1$,
$\frac{1}{z} - \frac{3}{y} - \frac{1}{x} + 7 = 0$.

18. $x + 2y + z = 3$,
$2x - y + 3z = 7$,
$3x + y + 4z = 10$.

19. $3x + 5y + 2z = 0$,
$12x - 15y + 4z = 12$,
$6x - 25y - 8z = 8$.

20. $2x - y + 4z = 3$,
$3x + 2y - 2z = -1$,
$x - 4y + 10z = 7$.

21. $2x + y + z - 3 = 0$,
$x + 4y + 3z - 10 = 0$,
$x - 3y - 2z + 7 = 0$.

22. $x - 2y - 3z = 2$,
$x - 4y - 13z = 14$,
$3x - 5y - 4z = 0$.

*23. We consider buying three kinds of food. Food I has one unit of vitamin A, three units of vitamin B, and four units of vitamin C. Food II has two, three and five units, respectively. Food III has three units each of vitamin A and vitamin C, none of vitamin B. We need to have 11 units of vitamin A, 9 of vitamin B, and 20 of vitamin C.

(a) Have we enough information to determine uniquely the amounts of each of the foods we must get?

(b) Suppose Food I costs 60 cents and the others 10 cents per unit. Is there a solution for this problem if exactly one dollar is spent for these foods?

*24. The solution set of the following system contains only one triple. Determine which of the equations may be omitted without altering the solution set.

$$\begin{cases} x + y = 5 \\ -x + 3z = 2 \\ x + 2y + z = 1 \\ y + z = -4 \end{cases}$$

---

*10. Equivalent Systems of Equations in Three Variables.

We give here a treatment of equivalent systems for first degree equations in three variables.

Recall the procedure used in Sections 7, 8, and 9 to study systems of first degree equations in three variables. We have been using the following operations which can always be performed upon the equations of a system to yield an equivalent system:

1. Two equations of the system may be interchanged.
2. An equation of the system may be multiplied by any number $k \neq 0$.
3. k times any equation of the system may be added to any other equation of the system.

Consider now the set of all equations that we can obtain from two given equations,

(10a) $$\begin{cases} x + 2y - z - 5 = 0 \\ x + y + z - 2 = 0 , \end{cases}$$

by multiplying the first equation by a constant, a, and the second equation by a constant, b (where a and b are not both zero), and then adding the two equations. This procedure involves operations (2) and (3). Thus, we can represent all such equations by

(10b) $$a(x + 2y - z - 5) + b(x + y + z - 2) = 0$$
(a, b not both zero).

By definition, any solution of the system (10a) must reduce each of the expressions in the parentheses in (10b) to zero. It must therefore be a solution of (10b).

For example, if we take $a = 2$, $b = 1$, we obtain

(10c) $$2(x + 2y - z - 5) + 1(x + y + z - 2) = 0$$
$$3x + 5y - z - 12 = 0 .$$

Since this equation is of first degree, it represents a plane. Since it is satisfied by all the triples in the solution set of (10a), the plane passes through the line of intersection of the planes in (10a). Hence, the equation (10c) represents a plane through the intersection of the planes in (10a).

Thus, any two distinct planes formed by substituting values of a and b in (10b) determine the same line of intersection as the equations in (10a). The left members of the equations obtained from (10b) are called linear combinations of the left members of the equations in (10a). We have used this converse proposition in Sections 7, 8, and 9.

Example 1: Find the equations of 2 distinct planes through the line of intersection of the planes of the system

$$y = 2$$
$$z = 5 .$$

Sketch the graph.

Solution: The general equation of all planes through the intersection of the given planes is equation

(10d) $$a(y - 2) + b(z - 5) = 0 \quad (a,b \text{ not both zero})$$

1. If we take $a = 1$, $b = 1$, we have

$$y - 2 + z - 5 = 0$$
$$y + z = 7 .$$

The given plane $y = 2$ is parallel to the XZ-plane and 2 units to the right of it. The given plane $z = 5$ is parallel to the XY-plane and 5 units above it. These planes intersect in a line parallel to the X-axis. The new plane $y + z = 7$ has the following traces:

In the XY-plane where $z = 0$,

$$y = 7 ;$$

in the YZ-plane where $x = 0$,

$$y + z = 7 ;$$

in the XZ-plane where $y = 0$,

$$z = 7 .$$

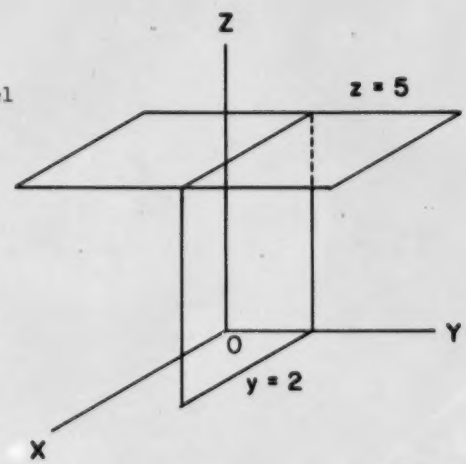


Figure 10a

It is a plane parallel to the X-axis. (See Figure 10b.) Note that the YZ-trace, $y + z = 7$, passes through the point $y = 2$, $z = 5$ in the YZ-plane.

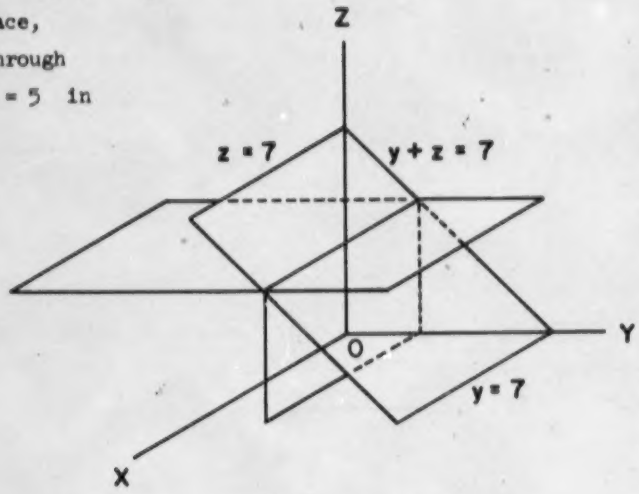


Figure 10b

2. If we take $a = 2$, $b = 2$ in Equation (10d) we have

$$2(y - 2) + 2(z - 5) = 0,$$
$$2y + 2z - 14 = 0.$$

This plane coincides with the plane we have just studied,

$$y + z = 7.$$

This is because the a and b we have chosen are both twice the a and b chosen above.

3. If we take $a = 2$, $b = 1$, we have

$$2(y - 2) + (z - 5) = 0$$
$$2y + z - 9 = 0.$$

The traces of this plane are

$$z = 0 \; ; \; y = \frac{9}{2} \; ;$$
$$x = 0 \; , \; 2y + z = 9 \; ;$$
$$y = 0 \; , \; z = 9.$$

This is another plane parallel to the X-axis. Notice again that the trace

$$2y + z = 9$$

passes through the point $(0,2,5)$. See Figure 10c.

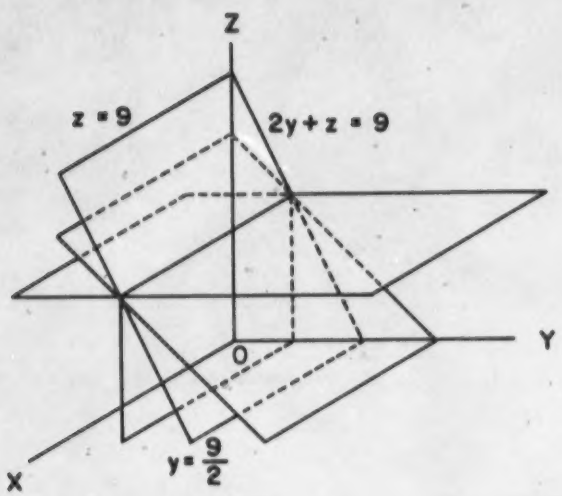


Figure 10c

Exercises 10.

1. Find an equation for a plane through the line of intersection of the planes in each of the following systems. By sketching the graph in each case, show that the plane represented by the equation you have found passes through the intersection of the given planes.

   (a) $x + 2 = 0$, $z - 4 = 0$.

   (b) $y + 4 = 0$, $z - 5 = 0$.

2. In each of the following problems, find an equation for the plane containing the given point and passing through the line of intersection of the given pair of planes.

   (a) (1,2,1) ; $x + 2y - 3z = 0$, $x - y + z = 1$.

   (b) (3,-1,0) ; $2y - 3z - 2 = 0$, $x + y + z = 0$.

   (c) The origin; $x + z = 0$, $2x - y + z - 8 = 0$.

   (d) (2,2,1) ; $2x - y + z - 3 = 0$, $x - 3y + 4 = 0$.

* 3. Prove that the planes represented by the equations

$$2x - y + 3z = 1$$
$$6x - 3y + 9z = 5$$

are parallel. Show that, for all values of a and b, both different from zero,

$$a(2x - y + 3z - 1) + b(6x - 3y + 9z - 5) = 0$$

represents a plane parallel to the given planes.

* 4. Find an equation for the plane containing the point $(1,-1,1)$ and passing through the line of intersection of the planes,

$$x + y - 3 = 0$$
$$z - 4 = 0 .$$

Sketch the graph, showing the traces of the three planes, and show that the plane represented by the equation you have found passes through the intersection of the given planes.

11. Miscellaneous Exercises.

1. A number may be written in the form $100h + 10t + u$, where h, t, and u represent respectively the hundreds, tens and units digits. If the sum of the digits of a certain number is 13, the sum of the units and tens digits is 10, and the number is increased by 99 if the digits are reversed, find the number.
2. Find the relation that must hold between the numbers a, b, c in order that the system

$$\begin{cases} 3x + 4y + 5z = a , \\ 4x + 5y + 6z = b , \\ 5x + 6y + 7z = c , \end{cases}$$

have a solution.
3. Find a three digit number such that the difference between each succeeding pair of digits is 1 and the sum of the digits is 15.
4. A man has three sums of money invested, one at 3%, one at 4%, and one at $4\frac{1}{2}\%$. His total annual income from the three investments is $\$346$. The first of these yields $\$44$ per year more than the other two combined. If all the money were invested at $3\frac{1}{2}\%$ he would receive $\$4$ per year more than he does now. How much is invested at each rate?

5. For what value of $\underline{a}$ will the three planes represented by the equations given below have a line of intersection? Give the coordinates of three points on the line.

$$x + y + z = 6$$
$$y - z = 1$$
$$2x - 3y + az = 7$$

6. Three trucks were hauling concrete. The first day one truck hauled 4 loads, the second hauled 3 loads, and the third hauled 5 loads. The second day the trucks hauled 5, 4, and 4 loads respectively. The third day the same trucks hauled 3, 5, and 3 loads respectively. If the trucks hauled 78 cu. yds. the first day, 81 cu. yds. the second day, and 69 cu. yds. the third day, find the capacity of each truck, assuming they were fully loaded on each trip.

7. Frank Nixon has a metal savings bank which registers the total amount deposited. Only pennies, nickels and dimes can be deposited. Frank knows that he has deposited one coin on each of 40 days. The bank shows a total deposit of \$1.80 . If Frank deposited as many pennies as both dimes and nickels, find the number of each.

8. A printing shop has three presses. One press operated 8 hours on Monday, 4 hours on Tuesday, and 2 hours on Wednesday. A second press operated 4 hours on Monday, 1 hour on Tuesday, and 5 hours on Wednesday. The third press operated 7 hours on Monday and 7 hours on Tuesday. Monday's output from the three presses was 1270 units, Tuesday's was 730 units, and Wednesday's was 550 units. What was the average output per hour for each press?

9. If A, B, C can do a piece of work in $2\frac{2}{3}$ days, A and B can do the work in $4\frac{4}{5}$ days, and C does twice as much work as A , at this rate, find the number of days in which each can do the work alone.

10. Three planes, A, B, C, working together can spray a certain cotton field in 2 hours. After they had worked together for one hour, plane C developed engine trouble, and planes A and B completed the job in one hour and 20 minutes more. The next day it was found necessary to respray the part sprayed by plane C . This was done by planes A and B in twenty minutes. How long would it take each plane to spray the entire field?

11. R, S, and T are the points of tangency of a triangle ABC circumscribed about a circle. If the sides of the triangle AB, BC, and AC are respectively 10, 8, and 7 units long, find the lengths of the segments AS, SB, BT, TC, CR, and AR.

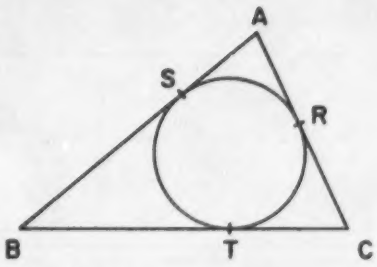


Figure 11a

12. If a parabola defined by the equation $y = ax^2 + bx + c$ passes through the points $(-1,1)$, $(3,1)$, $(4,-4)$, find the values of the constants a, b and c.

13. If a parabola defined by $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(-3,20)$, $(-1,0)$, find the values of the constants a, b, and c.

14. A local school gym entrance meter received half dollars from adults, quarters from high school pupils, and dimes from elementary school pupils. An attendant opened the box when the meter showed that 320 admissions had been deposited, giving a total of $76. He found there were twice as many dimes as quarters. Find the number of adults, high school pupils, and elementary school pupils who had paid admission.

15. The stopping distance of a car after the brakes are applied is given by the equation

$$s = \frac{1}{2}kt^2 + At + B$$

where

s = number of feet the car travels after the brakes are applied,

t = number of seconds the car is in motion after the brakes are applied.

If the following pairs of values were found for s and t, experimentally, find the values of the constants k, A, and B.

$\begin{cases} s = 46 \\ t = 1 \end{cases}$, $\begin{cases} s = 84 \\ t = 2 \end{cases}$, $\begin{cases} s = 114 \\ t = 3 \end{cases}$

16. Averages for a marking period in a certain mathematics class are based on scores made on a one-hour test, a short quiz, and a final examination. The scores made by Frank, Joyce, and Eunice, as well as their final averages, are shown in the following table.

	Test (T)	Quiz (Q)	Examination (E)	Final (A)
Frank	78	78	86	82
Joyce	78	98	74	80
Eunice	84	64	86	81

(a) Find values of w_1, w_2, w_3 that the instructor may have used to compute A if he used the formula

$$w_1T + w_2Q + w_3E = (w_1 + w_2 + w_3)A$$

to compute the final average, A.

(b) Can you find a triple of values for (w_1,w_2,w_3) whose sum is 1.

17. A firm sent a messenger to the post office to buy \$10 worth of 7¢ air mail stamps, 4¢ stamps and 1¢ stamps. The directions given were to buy as many air mail and 4¢ stamps as possible, getting twice as many air mail stamps as 4¢ stamps, and buying one cent stamps with the change that remained after the air mail and four cent stamps had been purchased. How many of each kind of stamps will the messenger obtain?

18. After playing 18 holes of golf, a player reports his score as a certain number. His actual score is 1 stroke per hole greater than the number which he reports. If the number which he gave as his score and his actual score are averaged the resulting number is $\frac{1}{5}$ greater than par. A score of 2 over par is less than the number he reports by 1. What is par for the course, and what number does he report as his score?

*19. Find an equation for the plane containing the points (-1,0,0), (1,-1,0), (-1,3,2).

20. [NOTE: This problem should interest students who have studied chemistry.] The problem of balancing chemical equations <u>can</u> be reduced to an easy algebraic process. We illustrate by several simple examples:

(a) Balance the equation for the following chemical reaction:

$$(\quad)FeS + (\quad)O_2 \longrightarrow (\quad)FeO + (\quad)SO_2$$

Insert the letters w, x, y, and z in the blanks and write down the equations resulting by equating the amounts of Fe in FeS and FeO.

$$(w)\ FeS + (x)\ O_2 \longrightarrow (y)\ FeO + (z)\ SO_2$$

$$w = y \qquad w(Fe) = y(Fe)$$

Repeat this process for the sulfur and oxygen.

$$w(S) = z(S) \quad ; \quad w = z$$

$$x(2\ O) = y(O) + z(2\ O)\ ;\ 2x = y + 2z$$

$$w = y$$

$$w = z$$

$$2x = y + 2z$$

Solve for x, y, and z in terms of w

$$y = w$$

$$z = w$$

$$x = \frac{3}{2}w$$

Choose w so that it is the smallest positive integer for which x, y, and z are also integers.

$$w = 2\ ; \qquad x = 3$$

$$y = 2\ ; \qquad z = 2$$

$$2\ FeS + 3\ O_2 \qquad 2\ FeO + 2\ SO_2\ .$$

(b) Balance the equation for the following chemical reaction:

$$(\quad)NH_3 + (\quad)O_2 \longrightarrow (\quad)H_2O + (\quad)NO_2$$

$$(a)NH_3 + (b)O_2 \longrightarrow (c)H_2O + (d)NO_2$$

Nitrogen: $a = d$

Hydrogen: $3a = 2c$

Oxygen: $2b = c + 2d$

$$d = a$$

$$c = \frac{3}{2}a$$

$$b = \frac{7}{4}a$$

$\therefore$ a must be equal to 4

$b = 7\ ;\ c = 6\ ;\ d = 4$

$$4\ NH_3 + 7\ O_2 \longrightarrow 6\ H_2O + 4\ NO_2$$

Balance the equations for the following chemical reactions.

(a) $Ag + HNO_3 \longrightarrow AgNO_3 + NO + H_2O$

(b) $AuCl_3 + KI \longrightarrow AuCl + KCl + I_2$

(c) $HNO_3 + HI \longrightarrow NO + I_2 + H_2O$

(d) $MnO_2 + HCl \longrightarrow MnCl_2 + Cl_2 + H_2O$

(e) $Cr(OH)_3 + NaOH + H_2O_2 \longrightarrow Na_2CrO_4 + H_2O$